THE UNIVERSITY OF TORONTO - MISSISSAUGA
Term Test 1 - Version 1 - Solutions
MAT135H5F - Fall 2020 - All sections

Time: 1 hour 50 minutes
Date: Friday October 9, 2020. 4:10PM - 6:00PM (EDT)
Instructors: N. Askaripour, J. Herman, H. Horowitz, J. de Jong, H. Lamei Ramandi, M. Pawliuk, S. Shen, J. Yang, Z. Wang.
Aids: Course notes, Course textbook, non-programmable calculator.

Submission

- **You must submit your completed solutions on Crowdmark by 6:00pm EDT Friday October 9, 2020**.
- Late submissions will not be accepted.
- You should start uploading your solutions no later than 5:40pm.
- If you require additional space, please insert extra pages.
- You do not need to print out this test; you may submit clear pictures/scans of your work on lined paper, or screenshots/PDFs of your work.

Additional Instructions

You must justify and support your solution to each question.

Permitted Resources

During the test:

- You may use any resources (course notes, textbook, videos) that have been posted to Quercus by instructors or TAs.
- You may use a non-programmable calculator.
- You may use personal notes related to official course material (from reading the textbook, participating in lectures/tutorials, posted course videos, completing WeBWork and Written Assignments).
- You may contact the instructors on Piazza using a private post.
- Do not use personal notes related to other material (e.g. notes created by studying external websites)
- Do not communicate with anyone other than the instructors.
- Do not use any online resources other than Piazza, Quercus, and Crowdmark.

Academic Integrity

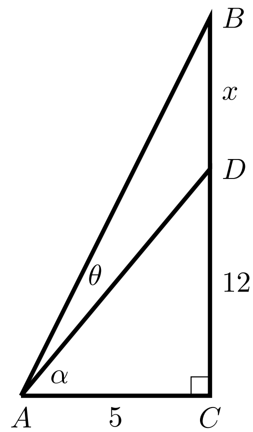
You should not discuss this test with anyone else while the test is happening.
By submitting this test I affirm that this test represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- I did not discuss this test with anyone during the test.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

Instructions: There are six (6) long answer questions worth five (5) points each, with multiple parts. Provide a complete solution, with justification.

Q1 (5 POINTS)

Let $\triangle ABC$ be a right triangle with $\angle BCA = \frac{\pi}{2}$ and $|AC| = 5$. D is a point on the segment BC and $|CD| = 12, |DB| = x$. Denote by $\theta = \angle BAD$, and $\alpha = \angle CAD$



(1) (3 pts) Show that $\theta = \arctan \frac{12+x}{5} - \arctan \frac{12}{5}$.

Solution. Note that since tan is $\dfrac{\text{opposite}}{\text{adjacent}}$ we get both:

$$\tan(\alpha) = \frac{12}{5} \qquad \text{and} \qquad \tan(\theta + \alpha) = \frac{x+12}{5}.$$

Therefore

$$\alpha = \arctan\left(\frac{12}{5}\right) \qquad \text{and} \qquad \theta + \alpha = \arctan\left(\frac{x+12}{5}\right).$$

Combining these gives

$$\theta = \arctan\left(\frac{x+12}{5}\right) - \alpha = \arctan\left(\frac{x+12}{5}\right) - \arctan\left(\frac{12}{5}\right),$$

as desired.

Grading. 1 point for using $\dfrac{\text{opposite}}{\text{adjacent}}$. 1 point for $\tan(\theta + \alpha) = \dfrac{x+12}{5}$. 1 point for a complete and clear argument.

(2) (2 pts) (**This is not related to part 1 or the above diagram**.) Evaluate $\sin(\arccos(-\frac{1}{3}))$. Your answer should be exact, not an approximation.

Solution. Let $\theta = \arccos(-\frac{1}{3})$. So $\cos\theta = \dfrac{-1}{3} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$.

A right triangle with angle θ, base -1 and hypotenuse 3 will have height $\sqrt{8}$ (by Pythagoras: $(-1)^2 + y^2 = 3^2$). Therfore $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{\sqrt{8}}{3}$.

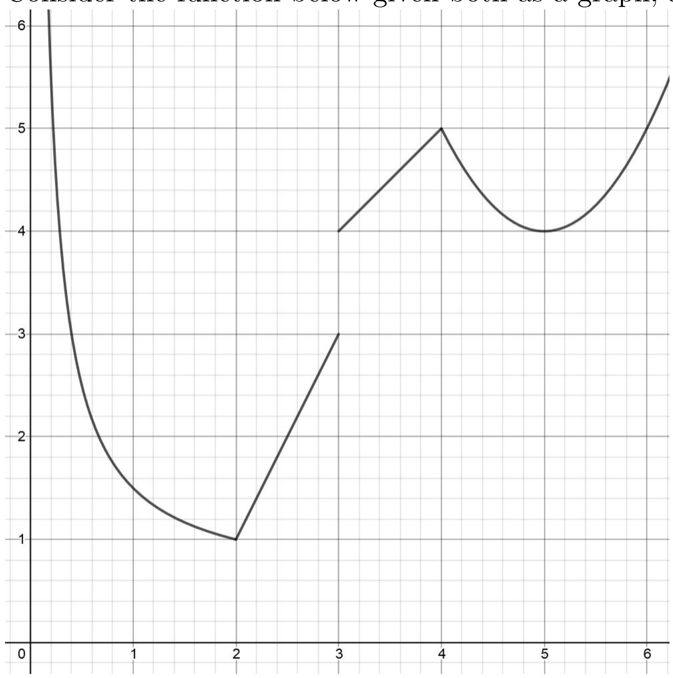
So $\sin(\arccos(-\frac{1}{3})) = \dfrac{\sqrt{8}}{3}$.

Grading. 0.5 point for drawing a relevant triangle, 0.5 for using Pythagoras, and 1 point for getting $\sin\theta$ as $\dfrac{\text{opposite}}{\text{hypotenuse}}$.

A correct final answer is not worth any points (since this can be evaluated from a calculator).

Q2 (5 POINTS)

Consider the function below given both as a graph, and as a piecewise function.



$$f(x) = \begin{cases} \frac{1}{x} + \frac{1}{2} & 0 < x < 2 \\ 2x - 3 & 2 \le x < 3 \\ x + 1 & 3 \le x < 4 \\ (x-5)^2 + 4 & x > 4 \end{cases}$$

(1) (2 pts) Find $\lim_{x \to 4} f(x)$, or justify why it does not exist.

Solution. From the diagram we see that this limit is 5.

Grading. This should be all or nothing. No justification is needed.

(2) (3 pts) Find $\lim_{x \to 3} f(f(x))$, or justify why it does not exist.

Solution. The left hand limit can be seen to be 3, by noting that for $x < 3$, but near 3, that $f(x)$ is slightly less than 3, so $f(f(x))$ is slightly less than 3.

The right hand limit can be seen to be 5, by noting that for $x > 3$, but near 3, that $f(x)$ is slightly more than 4, so $f(f(x))$ is slightly less than 5.

Since the left hand limit is not equal to the right hand limit, $\lim_{x \to 3} f(f(x))$ does not exist.

Grading. 1 point each for correctly computing the left and right hand limits (no explanation needed). 1 point for correctly observing that the two-sided limit does not exist because the individual limit do not exist.

Q3 (5 points)

Let $f(x) = \ln(\sqrt{x+1} + \sqrt{x})$ and $g(x) = \ln(\sqrt{x+1} - \sqrt{x})$.

(1) (3 pts) Find the domains of $f(x)$ and $g(x)$.

Solution. For both, because of the square roots, we must have $x \geq 0$.

For $f(x)$, because of the logarithm, we must have $\sqrt{x+1} + \sqrt{x} > 0$. This is always true since $x \geq 0$, and $\sqrt{x} \geq 0$, and $\sqrt{x+1} \geq 1$. So the domain of f is $[0, +\infty)$, or alternatively $x \geq 0$.

For $g(x)$, because of the logarithm, we must have $\sqrt{x+1} - \sqrt{x} > 0$. Equivalently, $\sqrt{x+1} > \sqrt{x}$, which is always true. So the domain of g is $[0, +\infty)$, or alternatively $x \geq 0$.

Grading. Each correct answer is worth 0.5 points. Using the square root restrictions is worth 1 point, and using the logarithm restrictions is worth 1 point.

(2) (2 pts) Show that $f(x) = -g(x)$.

Solution. Note that

$$\begin{aligned}
-g(x) &= -\ln(\sqrt{x+1} - \sqrt{x}) \\
&= \ln\left(\left(\sqrt{x+1} - \sqrt{x}\right)^{-1}\right) \\
&= \ln\left(\frac{1}{\sqrt{x+1} - \sqrt{x}}\right) \\
&= \ln\left(\frac{1}{\sqrt{x+1} - \sqrt{x}} \cdot \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}\right) \\
&= \ln\left(\frac{\sqrt{x+1} + \sqrt{x}}{x+1-x}\right) \\
&= \ln\left(\sqrt{x+1} + \sqrt{x}\right) = f(x).
\end{aligned}$$

Grading. 1 point for pulling the -1 inside the logarithm, 0.5 points for multiplying by the conjugate, and 0.5 for a full and complete argument.

Q4 (5 points)

Let $f(x) = \dfrac{1+2x}{3x-2}$.

(1) (2 pts) Show that $f(x)$ is one-to-one.

Solution. Suppose that $f(a) = f(b)$. We will show that $a = b$.

$$\begin{aligned} &f(a) = f(b) \\ \Rightarrow &\frac{1+2a}{3a-2} = \frac{1+2b}{3b-2} \\ \Rightarrow &(1+2a)(3b-2) = (1+2b)(3a-2) \\ \Rightarrow &3b + 6ab - 4a - 2 = 3a + 6ab - 4b - 2 \\ \Rightarrow &3b - 4a = 3a - 4b \\ \Rightarrow &7b = 7a \\ \Rightarrow &b = a. \end{aligned}$$

Grading. 1 point for the correct structure (Assume $f(a) = f(b)$ and try to show $a = b$.) and 1 point for a clear and complete argument

(2) (2 pts) Find a formula for $f^{-1}(x)$.

Solution. We exchange the role of x and y and then solve for x. So

$$\begin{aligned} &x = \frac{1+2y}{3y-2} \\ \Rightarrow &(3y-2)x = 1 + 2y \\ \Rightarrow &3yx - 2x = 1 + 2y \\ \Rightarrow &3yx - 2y = 1 + 2x \\ \Rightarrow &y = \frac{1+2x}{3x-2} \end{aligned}$$

So $f^{-1}(x) = \dfrac{1+2x}{3x-2}$. (This function has the unusual property that $f(x) = f^{-1}(x)$.)

Grading. 1 point for a good attempt at trying to solve for one variable in terms of the other. 1 point for a clear and complete argument.

(3) (1 pt) What are the domain and range of $f(x)$?

Solution. The only place where f and f^{-1} are not defined is when $3x - 2 = 0$. That is $x = \frac{2}{3}$. So the domains of these functions are $(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, +\infty)$ or equivalently "$x \neq \frac{2}{3}$".

Since $f(x) = f^{-1}(x)$ the range of f is the domain of f^{-1} (and vice versa). So the both have range $(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, +\infty)$.

Grading. 0.5 points for the domain, 0.5 points for the range. Both should have some kind of justification, but it doesn't have to be very in depth to get the point.

Q5 (5 points)

Let f and g be two quadratic functions whose graphs are translations of $y = x^2$.

(1) (2 pts) Suppose that $y = g(x)$ is symmetric about the line $x = 2$ and intersects the x-axis at $(0, 0)$. Find $g(x)$.

Solution. Since $g(x)$ is a translation of x^2, we know that there are parameters h, k so that $g(x) = (x - h)^2 + k$. (There are no stretchings or compressions.) Since g is symmetric about the line $x = 2$, we know that the vertex of g is at $x = 2$, so $h = 2$. Since g intersects the point $(0, 0)$, we know $0 = (0 - 2)^2 + k$. In other words, $0 = 4 + k$, and so $k = -4$. Therefore $g(x) = (x - 2)^2 - 4$.

Grading. 1 point for correctly determining h, and one point for correctly determining k. You may award half-points if the idea is correct, but the arithmetic was wrong.

(2) (3 pts) Suppose that $y = f(x)$ intersects the line $y = -3x$ at exactly one point $(-1, 3)$. Find $f(x)$.

Solution. Again, there are parameters h, k so that $f(x) = (x - h)^2 + k$. Since $f(x)$ intersects the point $(-1, 3)$ we know $3 = (-1 - h)^2 + k$. So $3 = 1 + 2h + h^2 + k$ and equivalently $\boxed{k = -h^2 - 2h + 2}$.
The second equation we find to relate h and k will come from setting the parabola and line equal.

$$\begin{aligned} &(x - h)^2 + k = -3x \\ \Rightarrow &x^2 - 2xh + h^2 + k = -3x \\ \Rightarrow &x^2 + (3 - 2h)x + (h^2 + k) = 0 \end{aligned}$$

Using the quadratic formula, we know that since there is exactly one intersection point, the discriminant ($b^2 - 4ac$) of this quadratic must be 0. So

$$\begin{aligned} &(3 - 2h)^2 - 4(1)(h^2 + k) = 0 \\ \Rightarrow &9 - 12h + 4h^2 - 4h^2 - 4k = 0 \\ \Rightarrow &\boxed{\text{9 - 12h - 4k = 0}} \end{aligned}$$

Putting our first boxed equation into this gives:

$$\begin{aligned} &9 - 12h - 4k = 0 \\ \Rightarrow &9 - 12h - 4(-h^2 - 2h + 2) = 0 \\ \Rightarrow &9 - 12h + 4h^2 + 8h - 8 = 0 \\ \Rightarrow &4h^2 - 4h + 1 = 0 \\ \Rightarrow &(2h - 1)^2 = 0 \\ \Rightarrow &2h - 1 = 0 \\ \Rightarrow &h = \frac{1}{2}. \end{aligned}$$

Substituting $h = \frac{1}{2}$ into $9 - 12h - 4k = 0$ gives:

$$\begin{aligned} &9 - 12\frac{1}{2} - 4k = 0 \\ \Rightarrow &9 - 6 = 4k \\ \Rightarrow &\frac{3}{4} = k \end{aligned}$$

Therefore $f(x) = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$.

Grading. 1 point for using $(-1, 3)$ to relate h and k. 1 point for correctly relating h and k by setting the functions equal. 1 point for a correct $f(x)$.

Q6 (5 points)

The online game 'Among Them' starts at its release date with 10 users, and every 5 days the number of users doubles.

How many days after the release date will it take for the number of Among Them users to be larger than the population of North America (which is about 600 million people)?

Solution. (This is very similar to Q2.1 on Written Assignment 1, the decibel question.)

Note that the formula that models this situation is $f(x) = 10 \cdot 2^{\frac{x}{5}}$, where x is the number of days after the release date. This is because the growth is doubling (exponential growth) so the base is 2, 10 corresponds to $f(0)$, and it doubles every 5 days (which is what the $\frac{x}{5}$ accounts for).

Let P be the population of North America.

We want to find the x where $P = 10 \cdot 2^{\frac{x}{5}}$. So

$$\begin{aligned}
&P = 10 \cdot 2^{\frac{x}{5}} \\
\Rightarrow &\frac{P}{10} = 2^{\frac{x}{5}} \\
\Rightarrow &\log_2\left(\frac{P}{10}\right) = \frac{x}{5} \\
\Rightarrow &x = 5 \cdot \log_2\left(\frac{P}{10}\right) \approx 129.2
\end{aligned}$$

So approximately 129.2 (or 130) days after the release date, the number of users will be larger than the population of North America.

Grading. The expression for $f(x)$ is worth 3 points: 1 point for using an exponential function of base 2, 1 point for correctly including the 10, and 1 point for correctly including the 5. Award half-points instead if no justification is given for the function.

2 points for a clear and correct argument for finding the number of days (using a logarithm of a different base is acceptable).

[End of Term Test 1]