THE UNIVERSITY OF TORONTO - MISSISSAUGA
Term Test 2 - Solutions
MAT135H5F - Fall 2020 - All sections

Time: 2 hours 10 minutes
Date: Friday November 27, 2020. 4:10PM - 6:20PM (EST)
Instructors: N. Askaripour, J. Herman, H. Horowitz, J. de Jong, H. Lamei Ramandi, M. Pawliuk, S. Shen, J. Yang, Z. Wang.
Aids: Course notes, Course textbook, non-programmable calculator.

Submission

- **You must submit your completed solutions on Crowdmark by 6:20pm EST Friday November 27, 2020**.
- Late submissions will not be accepted.
- You should start uploading your solutions no later than 5:40pm.
- If you require additional space, please insert extra pages.
- You do not need to print out this test; you may submit clear pictures/scans of your work on lined paper, or screenshots/PDFs of your work.

Additional Instructions

You must justify and support your solution to each question.

Permitted Resources

During the test:

- You may use any resources (course notes, textbook, videos) that have been posted to Quercus by instructors or TAs.
- You may use a non-programmable calculator.
- You may use personal notes related to official course material (from reading the textbook, participating in lectures/tutorials, posted course videos, completing WeBWork and Written Assignments).
- You may contact the instructors on Piazza using a private post.
- Do not use personal notes related to other material (e.g. notes created by studying external websites)
- Do not communicate with anyone other than the instructors.
- Do not use any online resources other than Piazza, Quercus, and Crowdmark.

Academic Integrity

You should not discuss this test with anyone else while the test is happening.
By submitting this test I affirm that this test represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- I did not discuss this test with anyone during the test.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters.*

Instructions: There are six (6) long answer questions worth five (5) points each, with multiple parts. Provide a complete solution, with justification.

Q1 (5 points)

Consider the function $f(x) = \dfrac{(4-x^2)}{(x-1)\sqrt{x+2}\sqrt{x+5}}$. **Note**: The same f is used in both parts of this question.

(1) (3 points) Find the vertical asymptotes of the function f.

Solution. The only possible vertical asymptotes are at $x = 1, -2$, and -5. Note

$$\lim_{x\to 1^+} \frac{(4-x^2)}{(x-1)\sqrt{x+2}\sqrt{x+5}} = \infty$$

therefore $x = 1$ is a vertical asymptote.

Checking $x = -2$:

$$\begin{aligned}\lim_{x\to -2^+} \frac{(4-x^2)}{(x-1)\sqrt{x+2}\sqrt{x+5}} &= \lim_{x\to -2^+} \frac{(2-x)\sqrt{x+2}\sqrt{x+2}}{(x-1)\sqrt{x+2}\sqrt{x+5}} \\ &= \lim_{x\to -2^+} \frac{(2-x)\sqrt{x+2}}{(x-1)\sqrt{x+5}} \\ &= 0\end{aligned}$$

Therefore $x = -2$ is not a vertical asymptote.

Since domain of f is $(-2, \infty)$ it is impossible to approach -5 through domain of f, so $x = -5$ is not a vertical asymptote for f.

Grading Scheme. [Grading]
0.5 point for clearly mentioning x=1 is vertical asymptote.

1 point for calculating limit of f as x approaches 1 (from right or from left, only one of them is enough), and finding the correct answer for the limit.

1 point for calculating lim of f as x approaches -2 from the right and show that the answer is 0 and conclude that $x = -2$ is not vertical asymptote.

0.5 for explaining why $x = -5$ is not a vertical asymptote (they do not need to find the domain, if the explanation is clear that it is impossible to approach -5 through a neighbourhood around -5).

Penalty:

-0.5 penalty if they write $x = -2$ is a vertical asymptote

-0.5 penalty if they write $x = -5$ is a vertical asymptote

-1.5 if there is nothing about $x = -2$ and $x = -5$

(2) (2 points) Find the horizontal asymptotes of the function f.

Solution. [Q1.2] Note that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{4}{x^2} - 1}{(1-\frac{1}{x})\sqrt{1+\frac{2}{x}}\sqrt{1+\frac{5}{x}}} = -1$$

(dividing top and bottom by x^2 and simplifying).
Therefore $y = -1$ is a horizontal asymptote.

It is impossible to approach $-\infty$ through domain of f.

Grading Scheme. [Grading]
1 point for setting the limit as $x \to \infty$ and evaluate it correctly.

0.5 for clearly mentioning $y = -1$ is horizontal asymp.

0.5 for explaining why we can not find another horizontal asymp. (-0.5 penalty for not mentioning that).

Q2 (5 points)

Let $f(x) = \begin{cases} \dfrac{1}{2x-3} & \text{if } x > 2 \\ -2x+5 & \text{if } 0 < x \leq 2 \\ x+5 & \text{if } x \leq 0. \end{cases}$

Note: The same f is used in both parts of this question.

(1) (3 points) Use first principles (the definition of the derivative) to show that for $x > 2$, $f'(x) = \dfrac{-2}{(2x-3)^2}$.

Solution. We use the definition of the derivative, obtaining

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2x-3+2h} - \frac{1}{2x-3}}{h} \\ &= \lim_{h\to 0} \frac{(2x-3)-(2x-3+2h)}{h(2x-3)(2x-3+2h)} \\ &= \lim_{h\to 0} \frac{-2}{(2x-3)(2x-3+2h)} \\ &= \frac{-2}{(2x-3)^2}. \end{aligned}$$

Solution. An alternate solution is possible using the definition (for $a > 2$):

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

Grading Scheme. 1 point for using the correct definition of the derivative, 2 for evaluating the limit.

(2) (2 point) Give all values of x for which $f(x)$ is not differentiable, and explain why.

Solution. The function is continuous. All three pieces are differentiable on their domains (two have constant derivative, the first is only not differentiable at $x = \frac{3}{2}$ which is not in its domain), so the only possible values where $f(x)$ is not differentiable are at $x = 0$ and $x = 2$. We have

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2x-3}\right)\Bigg|_{x=2} &= -2, \\ \frac{d}{dx}(-2x+5)|_{x=2} &= -2, \\ \frac{d}{dx}(-2x+5)|_{x=0} &= -2, \\ \frac{d}{dx}(x+5)|_{x=0} &= 1. \end{aligned}$$

Since the derivatives do not match at $x = 0$, $f(x)$ is not differentiable at $x = 0$.

Grading Scheme. 1 point for giving the correct value, and 1 point for a correct explanation. 1 point maximum if other values are given.

Q3 (5 points)

Note: For all parts of this question you may use any derivative rules and formulas from class without proof.

Point out exactly where you are using any derivative rules.

(1) (3 points) Find a point on the graph of $g(x) = \frac{2}{5}\sqrt{x^5}$ such that the tangent line to that point is parallel to the tangent line to $f(x) = \dfrac{2e^x}{3x+4}$ at $x = 0$.

Solution. First, by the quotient rule:

$$f'(x) = \frac{2e^x(3x+4) - 3(2e^x)}{(3x+4)^2}$$

So the slope of the tangent line to f at $x = 0$ is:

$$m = f'(0) = \frac{2e^0(4) - 3(2e^0)}{4^2} = \frac{2}{16} = \frac{1}{8}.$$

We must find a point $(x, g(x))$ such that $g'(x) = \dfrac{1}{8}$. Note that

$$g'(x) = \frac{2}{5}\frac{5}{2}x^{\frac{3}{2}} = x^{\frac{3}{2}} = \sqrt{x^3}.$$

So $\sqrt{x^3} = \frac{1}{8}$ implies $\sqrt{x} = \frac{1}{2}$ implies $x = \frac{1}{4}$.

Theferefore the desired point is $(\frac{1}{4}, g(\frac{1}{4})) = (\frac{1}{4}, \frac{1}{80})$

Grading Scheme. 0.5 point for finding $f'(x)$ correctly , 0.5 point for evaluating $f'(0)$ correctly.

0.5 point for finding $g'(x)$, 0.5 for setting $f'(0) = g'(x)$.

0.5 point for finding correct x , 0.5 for finding the point (x value and y value).

(2) (2 points) Let $f(x) = 2x^b$ and $g(x) = 3b^x$ for $b > 0$. Find b so that $f'(1) = g'(1)$.

Solution. First, by formulas from class:

$$f'(x) = 2bx^{b-1},$$
$$g'(x) = 3b^x(\ln b)$$

So at $x = 1$ we get

$$f'(1) = 2b,$$
$$g'(1) = 3b\ln(b).$$

So

$$\begin{aligned} & f'(1) = g'(1) \\ \implies & 2b = 3b\ln(b) \\ \implies & \ln(b) = \frac{2}{3} && \text{(since } b > 0 \text{ we can divide both sides by } b\text{)} \\ \implies & b = e^{\frac{2}{3}} \end{aligned}$$

Grading Scheme. 0.5 Evaluating $f'(1)$ (they must show $f'(x)$ to get the point) , 0.5 evaluating $g'(1)$ (they must show $g'(x)$ for this step).

1 point setting the equation and solving it correctly.

Penalty: -0.5 if they do not explain why it is ok to divide both sides by b.

Q4 (5 points)

Note: For all parts of this question you may use any derivative rules and formulas from class without proof.

Point out exactly where you are using any derivative rules.

(1) (1 point) Let $f(x) = \tan(\sin(x^2))$. Compute $f'(x)$.

Solution. Note that

$$\begin{aligned} f'(x) &= \sec^2(\sin(x^2)) \cdot \left(\sin(x^2)\right)' && \text{(chain rule)} \\ &= \sec^2(\sin(x^2))\cos(x^2)(x^2)' && \text{(chain rule)} \\ &= \sec^2(\sin(x^2))\cos(x^2)2x && \text{(power rule)} \end{aligned}$$

Grading Scheme. 0.5 points for the tan derivative, and 0.5 for correctly using the chain rule.

(2) (2 points) Calculate $h'(1)$ where h is the inverse of $g(x) = x + x^3 + \cos(x)$.

Solution. Note that $g(0) = 0 + 0^3 + \cos(0) = 1$. So $h(1) = 0$. Also $g'(x) = 1 + 3x^2 - \sin x$. By the formula for the derivative of an inverse we have:

$$h'(1) = \frac{1}{g'(h(1))} = \frac{1}{g'(0)} = \frac{1}{1+0-0} = 1.$$

Grading Scheme. 1 point for realizing $h(1) = 0$, and 1 point for correctly using the derivative of the inverse formula.

(3) (2 points) Let $xy = \cos(y) + x^2$. Compute the tangent line to this curve at $(0, \frac{\pi}{2})$.

Solution. Using implicit differentiation we get:

$$\begin{aligned} & xy = \cos(y) + x^2 \\ \implies & \frac{d}{dx}(xy) = \frac{d}{dx}(\cos(y) + x^2) \\ \implies & y + x\frac{dy}{dx} = -\sin(y)\frac{dy}{dx} + 2x && \text{(Product rule, chain rule, power rule)} \\ \implies & x\frac{dy}{dx} + \sin(y)\frac{dy}{dx} = 2x - y \\ \implies & \frac{dy}{dx} = \frac{2x - y}{x + \sin(y)} \end{aligned}$$

Let $y = mx + b$ be the tangent line to the curve $xy = \cos(y) + x^2$ at $(0, \frac{\pi}{2})$. We get the slope m by plugging it into the derivative formula we just computed:

$$m = \frac{2(0) - \frac{\pi}{2}}{0 + \sin(\frac{\pi}{2})} = \frac{-\frac{\pi}{2}}{1} = \frac{-\pi}{2}.$$

Now the line passes through the point $(0, \frac{\pi}{2})$, so we get:

$$\frac{\pi}{2} = m(0) + b = b$$

Therefore the tangent line is $y = \frac{-\pi}{2}x + \frac{\pi}{2}$.

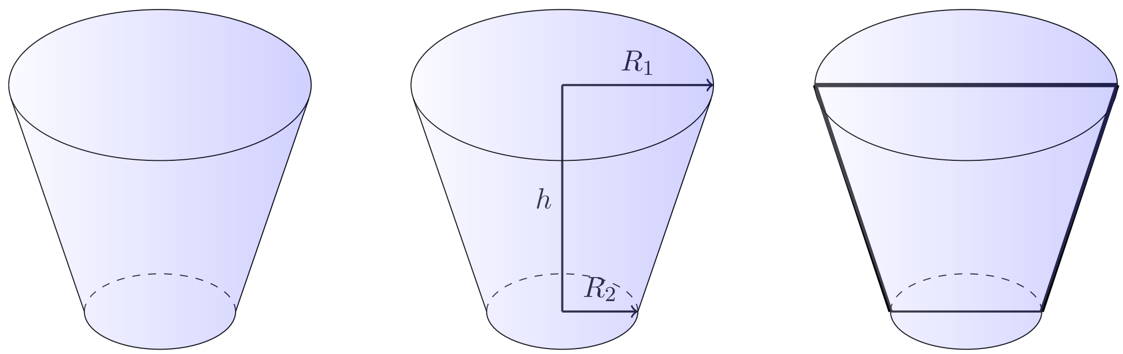
Grading Scheme. 1 point for correct implicit differentiation, 1 point for finding the correct slope (based on the derivative they computed), 1 point for the correct y-intercept (based on their computed work).

Q5 (5 points)

Note: For all parts of this question you may use any derivative rules and formulas from class without proof.

Point out exactly where you are using any derivative rules.

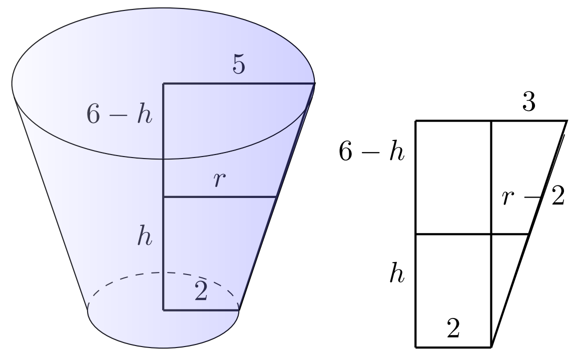
A person is pouring coffee into a cup at rate of $3\text{cm}^3/\text{min}$. The bottom and the top of the cup are circles with radii (respectively) 2cm. and 5cm. The height h of the cup (the line which joins the centers of the top circle and the bottom circle) is 6cm (see center image). The two parallel radii on the top circle and the bottom circle, the line joining the centers, and the line on the side joining the end point of the considered radii make a trapezoid (see rightmost image).



What is the rate of change of the height of the coffee when the radius of poured coffee in the cup is 4cm?

Hint. $V = \frac{1}{3}\pi h(R_1^2 + R_1R_2 + R_2^2)$ is the formula for volume of such a cup of height h, and with the top circle having radius R_1 and the bottom circle having radius R_2.

Solution. Solution 1, using a triangle in the trapezoid Let $h = h(t)$ be the height of the coffee at a given time and let $r = r(t)$ be the radius of the coffee at a given time.



Things we know: The volume $V = V(t)$ of the coffee at a given time is $V = \frac{\pi}{3}h(r^2+2r+4)$, which we know from the formula given in the hint (using the cup's bottom radius of 2 cm.) We also know $\frac{dV}{dt} = 3\text{cm}^3/\text{min}$.

Things we want: We want to know $\frac{dh}{dt}$ at $r = 4$.

Relating h and r. Using similar triangles we get $\dfrac{h}{r-2} = \dfrac{6}{3}$. This simplifies to $r = \dfrac{h+4}{2}$. By rearranging we can see that when $r = 4$, then $h = 4$.

Getting V as a function of h. Plugging in r into our formula for V gives:

$$\begin{aligned} V &= \frac{\pi}{3}h(r^2+2r+4) \\ &= \frac{\pi}{3}h(\left(\frac{h+4}{2}\right)^2 + 2\left(\frac{h+4}{2}\right) + 4) \\ &= \frac{\pi}{3}h(\frac{h^2+8h+16}{4} + h + 4 + 4) \\ &= \frac{\pi}{3}h(\frac{h^2+8h+16}{4} + \frac{4h+32}{4}) \\ &= \frac{\pi}{3}h(\frac{h^2+12h+48}{4}) \\ &= \frac{\pi}{12}(h^3+12h^2+48h) \end{aligned}$$

Differentiating. Differentiating both with respect to time gives:

$$\frac{dV}{dt} = \frac{\pi}{12}(3h^2\frac{dh}{dt} + 24h\frac{dh}{dt} + 48\frac{dh}{dt}) = \frac{dh}{dt}\frac{\pi}{4}(h^2+8h+16).$$

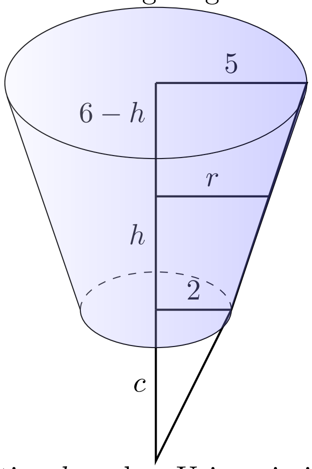
So

$$\frac{dh}{dt} = \frac{4}{\pi(h^2+8h+16)}\frac{dV}{dt}.$$

At $r = 4$ we have $h = 4$ and $\frac{dV}{dt} = 3$, so we get

$$\left.\frac{dh}{dt}\right|_{h=4} = \frac{4}{\pi(4^2+8(4)+16)}3 = \frac{12}{64\pi} = \frac{3}{16\pi} \approx 0.060\text{cm/min}$$

Solution. Solution 2, using a cone Let $h = h(t)$ be the height of the coffee at a given time and let $r = r(t)$ be the radius of the coffee at a given time. We will also find it useful to think about triangles instead of trapezoids, so we will call c (which does not depend on time), the distance needed to continue along below the height to make a cone (or triangle). See the following diagram:



Things we know: The volume $V = V(t)$ of the coffee at a given time is $V = \frac{\pi}{3}h(r^2 + 2r + 4)$, which we know from the formula given in the hint (using the cup's bottom radius of 2 cm.) We also know $\frac{dV}{dt} = 3\text{cm}^3/\text{min}$.

Things we want: We want to know $\frac{dh}{dt}$ at $r = 4$.

What is c? Using similar triangles (on the largest and smallest triangles) we get $\frac{6+c}{5} = \frac{c}{2}$. Isolating for c (which takes a couple steps) gives $c = 4$.

Relating h and r. Using similar triangles (on the middle and smallest triangles, knowing $c = 4$) we get $\frac{h+4}{r} = \frac{4}{2}$. This simplifies to $r = \frac{h+4}{2}$. By rearranging we can see that when $r = 4$, then $h = 4$.

Getting V as a function of h. Plugging in r into our formula for V gives:

$$\begin{aligned}
V &= \frac{\pi}{3}h(r^2 + 2r + 4) \\
&= \frac{\pi}{3}h(\left(\frac{h+4}{2}\right)^2 + 2\left(\frac{h+4}{2}\right) + 4) \\
&= \frac{\pi}{3}h(\frac{h^2 + 8h + 16}{4} + h + 4 + 4) \\
&= \frac{\pi}{3}h(\frac{h^2 + 8h + 16}{4} + \frac{4h + 32}{4}) \\
&= \frac{\pi}{3}h(\frac{h^2 + 12h + 48}{4}) \\
&= \frac{\pi}{12}(h^3 + 12h^2 + 48h)
\end{aligned}$$

Differentiating. Differentiating both with respect to time gives:

$$\frac{dV}{dt} = \frac{\pi}{12}(3h^2\frac{dh}{dt} + 24h\frac{dh}{dt} + 48\frac{dh}{dt}) = \frac{dh}{dt}\frac{\pi}{4}(h^2 + 8h + 16).$$

So

$$\frac{dh}{dt} = \frac{4}{\pi(h^2 + 8h + 16)}\frac{dV}{dt}.$$

At $r = 4$ we have $h = 4$ and $\frac{dV}{dt} = 3$, so we get

$$\left.\frac{dh}{dt}\right|_{h=4} = \frac{4}{\pi(4^2 + 8(4) + 16)}3 = \frac{12}{64\pi} = \frac{3}{16\pi} \approx 0.060\text{cm/min}$$

Grading Scheme.

- 1 point for a good start (e.g. sorting into haves/wants)
- 1 point for correctly writing the volume formula in terms of V, h, r.
- 1 point for finding r in terms of h.
- 1 point for correctly differentiating V as a function of h.
- 1 point for a clear (not necessarily correct) final answer in the interval $[0.001, 10]$

Q6 (5 POINTS)

Note: For all parts of this question you may use any derivative rules and formulas from class without proof.

Point out exactly where you are using any derivative rules.

(1) (2 points) Let

$$L = \lim_{x \to 0} \frac{\cos(x) - 1}{x^2 + f(x)}.$$

For each of the following choices of $f(x)$, compute L using l'Hopital's Rule, or explain why l'Hopital's Rule cannot be used.

a) $f(x) = x^3$,

Solution. Since $\lim_{x \to 0}(\cos(x) - 1) = 0$ and $\lim_{x \to 0}(x^2 + x^3) = 0$, this is an indeterminate form of type $\frac{0}{0}$. We apply l'Hopital's Rule, obtaining

$$L = \lim_{x \to 0} \frac{\cos(x) - 1}{x^2 + x^3} = \lim_{x \to 0} \frac{-\sin(x)}{2x + 3x^2}.$$

Now $\lim_{x \to 0}(-\sin(x)) = 0$ and $\lim_{x \to 0}(2x + 3x^2) = 0$, so this is also an indeterminate form of the type $\frac{0}{0}$. We apply l'Hopital's Rule again, obtaining

$$L = \lim_{x \to 0} \frac{-\sin(x)}{2x + 3x^2} = \lim_{x \to 0} \frac{-\cos(x)}{2 + 6x}.$$

Now $\lim_{x \to 0}(-\cos(x)) = -1$, and $\lim_{x \to 0}(2 + 6x) = 2$, so $L = \frac{-1}{2}$.

Grading Scheme. 0.5 for identifying the type and that we can use l'Hopital's Rule. 0.5 for evaluating the limit.

b) $f(x) = \frac{1}{x^2}$.

Solution. We have $\lim_{x \to 0}(\cos(x) - 1) = 0$ and $\lim_{x \to 0} x + \dfrac{1}{x^2} = \infty$, so this limit is of the form $\frac{0}{\infty}$. This is not an indeterminate form (it gives a limit of zero without further manipulation), and thus l'Hopital's Rule does not apply.

Grading Scheme. 0.5 for identifying the type. 0.5 for explaining why l'Hopital's Rule doesn't apply. (It is not necessary to evaluate the limit.)

(2) (3 points) Let $g(x) = x^3 - 5x + 1$. Find the absolute maxima and absolute minima of $g(x)$ on $[0, 2]$. (You should give us both the x-values and the y-values where these occur).

Solution. We first find the critical point(s) of $g(x)$ by setting the derivative to be zero. This yields the equation

$$0 = g'(x) = 3x^2 - 5.$$

Solving this gives critical values $x = \pm\sqrt{\frac{5}{3}}$. Only the positive square root is in the domain. We find the value of $g(x)$ at the critical value and the endpoints of the domain. These are

$$g(0) = 1,$$
$$g(2) = 8 - 10 + 1 = -1,$$
$$g\left(\sqrt{\frac{5}{3}}\right) = \frac{5}{3}\sqrt{\frac{5}{3}} - 5\sqrt{\frac{5}{3}} + 1 = 1 + \frac{5\sqrt{5} - 15\sqrt{5}}{3\sqrt{3}} = 1 - \frac{10\sqrt{5}}{3\sqrt{3}} < -1.$$

Therefore $\left(\sqrt{\frac{5}{3}}, 1 - \frac{10\sqrt{5}}{3\sqrt{3}}\right)$ is the absolute minimum, and $(0, 1)$ is the absolute maximum.

Grading Scheme. 1 point for finding the critical value, 1 for checking the value at the critical and end points, and 0.5 for each correct conclusion.

[End of Term Test 2]